Discrete Mathematics

Relations

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- domain, codomain, image, preimage
- inverse and composition
- properties of relations
- closure of relation
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Binary relation

Let A, B be two sets. A **binary relation** between the elements of A and B is any subset of the Cartesian product of A and B, i.e. $R \subseteq A \times B$.

We denote relations by capital letters, e.g. R, S, etc.

We say that two elements $a \in A$ and $b \in B$ are in relation R iff the pair $(a, b) \in R$ (it can be also denoted as: aRb).

Examples

- **empty relation** (no pair belongs to it)
- **diagonal** relation $\Delta = \{(x, x) : x \in X\}$ (it is the "equality" relation)
- **full relation**: any pair belongs to it (i.e. $R = X^2$)

Binary relation as a predicate and as a graph

Binary relation can be represented as a predicate with 2 free variables as follows:

Given a predicate $R(x, y)$, for $x \in X$ and $y \in Y$, the relation is the set of all pairs $(x, y) \in X \times Y$ that satisfy the predicate (i.e. make it true)

Each binary relation can be naturally represented as a graph.

Example

$R(x, y)$: “x is less than y”

The relation R represented by the above predicate is the set of all pairs $(x, y) \in X \times Y$ so that $R(x, y)$ is true (i.e. $x < y$)

Examples of binary relations

$A = B = \mathcal{N}$

- diagonal relation Δ $(x = y)$
- $x > y$
- $x \leq y$
- x is a divisor of y
- x and y have common divisor
- $x^2 + y^2 \geq 10$

More examples

Examples of relations on the set $P \times P$, where P is the set of all people.

- $(x, y) \in R \Leftrightarrow$ x is a son of y
- $(x, y) \in R \Leftrightarrow$ x is the mother of y
- $(x, y) \in R \Leftrightarrow$ x is the father of y
- $(x, y) \in R \Leftrightarrow$ x is a grandmother of y

More examples

Examples of $R \subseteq P \times C$, where C is the set of all courses in the univeristy for last 5 years.

- $(p, c) \in R \Leftrightarrow$ p passed course c
- $(p, c) \in R \Leftrightarrow$ p attended course c
- $(p, c) \in R \Leftrightarrow$ p thinks course c is interesting

Domain and co-domain of relation

For binary relation $R \subseteq A \times B$, the set A is called its **domain** and B is called its **co-domain**

Domain and co-domain can be the same set.

Image and pre-image of relation

Pre-image of binary relation $R \subseteq X \times Y$:
$\{x \in X : \exists y \in Y (x, y) \in R\}$

Image of binary relation $R \subseteq X \times Y$:
$\{y \in Y : \exists x \in X (x, y) \in R\}$

Example

$A = \{1, 3, 5, 6\}$, $B = \{3, 4, 5, 6, 7\}$. Relation $R \subseteq A \times B$ is defined as follows:

$xRy \Leftrightarrow x > y$
$R =?$

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domain of R?:

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$R =?$ $\{(5, 3), (5, 4), (6, 3), (6, 4), (6, 5)\}$
domain of R?: A
co-domain of R?:

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domain of R?: A
co-domain of R?: B
pre-image of R?: $\{5, 6\}$
image of R?: $\{3, 4, 5\}$

Inverse of relation

If $R \subseteq X \times Y$ is a binary relation then its **inverse** $R^{-1} \subseteq Y \times X$ is defined as $R^{-1} = \{(y, x) : (x, y) \in R\}$

Examples: what is the inverse of:
"x < y"?

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Examples: what is the inverse of:
"x < y"?
"x is a parent of y"?

Composition of relations

If $S \subseteq A \times B$ and $R \subseteq B \times C$ are two binary relations on sets A,B and B,C, respectively, then the **composition** of these relations, denoted as $R \circ S$ is the binary relation defined as follows:

$$R \circ S = \{(a, c) \in A \times C : \exists_{b \in B}[(a, b) \in R \land (b, c) \in S]\}$$

Sometimes it is denoted as RS. If $R = S$ then the composition of R with itself: $R \circ R$ can be denoted as R^2.

More than 2 relations can be composed. We denote the n-th composition of R with itself as R^n (e.g. $R^3 = R \circ R \circ R$, etc.)

Composition is associative, i.e.:
$(R \circ S) \circ T = R \circ (S \circ T)$

Example

$A = \{0, 1, 2, 3, 4\}$, $B = \{a, b, c\}$, $C = \{x, y, z, v\}$

$R = \{(1, a), (2, c), (3, a)\}$,
$S = \{(a, z), (a, v), (b, x), (b, z), (c, y)\}$

$R \circ S = ?$

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For what binary relations their composition is commutative?

Properties

The following abstract properties of binary relations are commonly used:

- reflexivity
- symmetry
- counter-symmetry
- anti-symmetry
- transitivity
- connectedness

Reflexivity

Binary relation $R \subseteq X \times X$ is **reflexive** iff:

$\forall x \in X \; xRx$

Examples? (assume X is the set of all positive naturals)

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diagonal relation Δ (i.e. $x == y$)?

Symmetry

Binary relation $R \subseteq X \times X$ is **symmetric** iff:

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"x and y have common divisor"?
$x \leq y$?
"x is a divisor of y" ?

Transitivity

Binary relation $R \subseteq X \times X$ is **transitive** iff:

$\forall (x, y, z) \in X, xRy \land yRz \Rightarrow xRz$

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Closure of a relation

A **closure** of a binary relation R with regard to (wrt) some property P is the binary relation S such that the following conditions hold:

- S has the property P
- $R \subseteq S$ (S "extends" R)
- S is the smallest (with regard to inclusion) relation satisfying the two above conditions (i.e. for any T such that $R \subseteq T$ it holds that $S \subseteq T$.

The property P can be for example: transitivity, symmetry, reflexivity, etc.

Notice: the closure of relation **may not exist** (example?:

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Notice: the closure of relation **may not exist** (example?: a counter-symmetric closure of a symmetric relation, etc.)

Examples: How to compute the closure of a relation?

If R is a binary relation, lets consider how to compute its reflexive, symmetric and transitive closure:

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- reflexive closure of R? $R \cup \Delta$
- symmetric closure of R? $R \cup R^{-1}$
- transtitive closure of R? $R \cup R^2 \cup R^3 ... = \bigcup_{i \in N^+} R^i$

Examples: transitive closure of relation

For a binary relation $R \subseteq X^2$ its **transitive closure** is defined as the smallest relation T so that T is transitive and $R \subseteq T$

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Example: transitive closure of:
"x is a son of y"?
"x == y"?
"$x \geq y$"?

Equivalence relation

A binary relation $R \subseteq X^2$ is **equivalence relation** iff it is:

- reflexive
- symmetric
- transitive

Examples

Examples? (assume X is the set of all positive naturals)

Examples

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$x == y$?

Examples

Examples? (assume X is the set of all positive naturals)

$x == y$?
"x and y have common divisor"?

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Examples? (assume X is the set of all positive naturals)

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"x and y have common divisor"?
$x \leq y$?

Examples

Examples? (assume X is the set of all positive naturals)

$x == y$?
"x and y have common divisor"?
$x \leq y$?
"x-y is even"?

Equivalence class

An **equivalence class** of the element $x \in X$ of the equivalence relation $R \subseteq X^2$ is defined as:

$$[x]_R = \{y \in X : xRy\}$$

(notice that, due to symmetry of equivalence relation, xRy is equivalent to yRx)

For $[x]_R$, x is called the **representative** of this equivalence class.

There can be many representatives of the same equivalence class.

Partition of a set

A family F of non-empty subsets of some set X is called **partition** of X if the following two conditions hold:

- for any two different $A, B \in F$ it holds that $A \cap B = \emptyset$
- X is the union of all sets from F ($X = \bigcup F$)

Each set from F is called a **partition block**.

Examples?

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Examples?
odd an even numbers form two blocks of partition of integers

Properties of equivalence classes

If $[x]_R$ and $[y]_R$ are two equivalence classes of some equivalence relation R, then either:

- $[x]_R \cap [y]_R = \emptyset$ (do not intersect)
  or:
- $[x]_R == [y]_R$ (are identical)

Since $\forall x \in X \; [x]_R \neq \emptyset$ (due to reflexivity of R), and different equivalence classes are disjoint the following holds:

The equivalence classes **partition** the domain of the equivalence relation.

Example

What are the equivalence classes of the following equivalence relations?

- x == y
- "x has the same diploma supervisor as y"

Quotient of the set by equivalence relation R (operation of abstraction)

Given an equivalence relation $R \subseteq X^2$ we call the family of all its equivalence classes the **quotient of X by R**:

$X/R = \{[x]_R : x \in X\}$

(the similarity to division symbol for numbers is not coincidental, since it has some similar properties)

The X/R operation is also called the "abstraction operation", i.e. we abstract from any properties that are indifferent for the equivalence relation R.

Example

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What is X/R if:

- X is the set of natural numbers and R is equality $(x = y)$?
- P is the set of students and R is the set of pairs of students that have the same diploma supervisor?

Order

Consider a relation $R \subseteq X^2$ is called a **partial order** and four properties:

1. reflexive
2. anti-symmetric
3. transitive
4. $\forall x, y \in X \; xRy \vee yRx$

Relation R is:

- partial order if it satisfies conditions 1-3 above
- quasi order if it satisfies only 1 and 2
- linear order if it satisfies all conditions 1-4 above

Examples

Is the following relation a partial order, quasi order, linear order, ?

- $\leq$ (on numbers) ?
- Δ (on any set)? ("x=y")
- $<$ (on numbers)
- $\subseteq$ (on sets)?

Generalisation: n-ary relation

An n-ary relation R, for $n \in \mathcal{N}$ is defined as $R \subseteq X_1 \times X_2 \ldots X_n$.
Binary relation is a special case for $n = 2$.
In particular, for:

- $n = 1$, 1-ary relation is the set of some elements of the domain that satisfy some property (e.g. even numbers, etc.)
- $n = 0$, 0-ary relation, that is empty can be theoretically interpreted as a *constant* in the domain of the relation (e.g. "0" in natural numbers) that has some special properties

Example tasks/questions/problems

For each of the following: precise definition and ability to compute on the given example (if applicable):

- Relation and basic concepts
- Properties of binary relations
- Composition and inverse
- Equivalence relation, equivalence classes

Thank you for your attention.