Discrete Mathematics

Mathematical Induction

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Contents

- Mathematical Induction
- Examples of numerical and non-numerical statements than can be proven by mathematical induction
- Strong Mathematical Induction
- Recursive definitions
- Equivalence of Mathematical Induction with the well-ordering of the natural numbers

Statements about Natural Numbers

Imagine a **statement** concerning all natural numbers greater than some natural value that can be expressed in the form of a predicate:

$$\forall_{n \geq n_0} P(n)$$

where $n \in N$ is a free natural variable, and n_0 is the smallest value having the property

Examples of $\forall_{n>n_0} P(n)$:
"for any $n \geq 0$ it holds that $n < 2^n$"
"for any $n \geq 0$ the sum of first n odd numbers is equal to n^2"
"for any $n \geq 1$ it holds that $2^n < n!$"

Mathematical Induction

The principle of mathematical induction:

If the following 2 conditions hold, for some predicate $P(n)$, $n \in N$:

1. $P(n_0)$ is true for some $n_0 \in N$ (**Basis step**)
2. $P(k) \Rightarrow P(k+1)$ is true for any $k \geq n_0$ (**Inductive step**)[1]

then: the predicate $P(n)$ is true for all $n \geq n_0$.

Mathematical Induction is a **powerful technique** for proving statements concerning natural numbers of the form $\forall_{n \geq n_0} P(n)$.

[1] $P(k)$ is called "inductive assumption"

Sum notation (reminder)

Let a_i be a sequence of numbers indexed by natural index $i \in N$. Then notation:

$$\sum_{i=i_0}^{k} a_i$$

Where:

- i is the name of the index variable
- a_i is a sequence of numbers indexed by i

Denotes **the sum** of all the terms of the sequence a_i from a_{i_0} up to a_k (both inclusive):

$$\sum_{i=i_0}^{k} a_i = a_{i_0} + \cdots + a_k$$

Examples of sum notation

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$\sum_{i=2}^{5} i =$

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$\sum_{i=2}^{5} i = 2 + 3 + 4 + 5$

$\sum_{i=4}^{6} i^2 =$

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$\sum_{i=2}^{5} i = 2 + 3 + 4 + 5$

$\sum_{i=4}^{6} i^2 = 4^2 + 5^2 + 6^2 = 16 + 25 + 36 = 77$

Product notation

Let a_i be a sequence of numbers indexed by natural index $i \in N$. Then notation:

$$\prod_{i=i_0}^{k} a_i$$

Where:

- i is the name of the index variable
- a_i is a sequence of numbers indexed by i

Denotes **the product** of all the terms of the sequence a_i from a_{i_0} up to a_k (both inclusive):
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Example:

$\prod_{i=1}^{n} i =$

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Example:

$\prod_{i=1}^{n} i = 1 \cdot 2 \cdot \ldots \cdot n = n!$

Triangle Numbers

$$P(n):$$

$$T_n = \sum_{i=1}^{n} i =$$

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The sum of n first non-negative natural numbers is called **triangle number**.

Is the above equation true for all $n \in N$?

(proof by mathematical induction)

Proof of the formula for Triangle Numbers

Basis step:

$P(1)$:

- left-hand side: $\sum_{i=1}^{1} = 1$
- right-hand side: $1 \cdot (1+1)/2 = 1$

$P(1)$ holds (i.e. the basis step is done)

Inductive assumption:

$\forall_{k \geq 1} \sum_{i=1}^{k} i = k(k+1)/2$

Inductive step (the main part of the proof):

$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = k(k+1)/2 + (k+1) =$
$k(k+1)/2 + 2(k+1)/2 = (k+2)(k+1)/2$

The above is equivalent to $P(k+1)$, so that the inductive step is done, what completes the proof for all $n > 0$.

Sum of geometric series

$a, r \in R,\ r \neq 1$

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$$\sum_{i=0}^{n} ar^i =$$

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$$a + ar + ar^2 + \ldots + ar^n = ?$$

Sum of geometric series

$a, r \in R,\ r \neq 1$

$$P(n):$$

$$\sum_{i=0}^{n} ar^i =$$

$$a + ar + ar^2 + \ldots + ar^n = ?$$

$$= \frac{ar^{n+1} - a}{r - 1}$$

Is the above equation true for all $n \in N$?

(proof by mathematical induction)

Geometric series formula proof

Basis step:
$P(0)$:

- left-hand side: $\sum_{i=0}^{0} ar^i = ar^0 = a \cdot 1 = a$
- right-hand side: $(ar^{0+1} - a)/(r-1) = (r-1)a/(r-1) = a$

the basis step is done.
Inductive assumption: $\sum_{i=0}^{k} ar^i = \frac{ar^{k+1}-a}{r-1}$
Inductive step:
$\sum_{i=0}^{k} ar^i = \frac{ar^{k+1}-a}{r-1} + ar^{k+1} = \frac{ar^{k+1}-a}{r-1} + \frac{ar^{k+2}-ar^{k+1}}{r-1} = \frac{ar^{k+2}-a}{r-1}$
The inductive step is done what completes the proof.

Recursive Definition

Mathematical Induction makes it also possible to define some mathematical objects indexed by natural numbers in a **recursive** way i.e. the defined object references to itself but for a smaller natural value and some *basis* object is defined. Recursive definition constists of two parts:

1. basis case
2. recursive (inductive) step

Example of recursive definition

Factorial of n:

Denoted as: $n!$

It is a product of n first non-zero natural numbers.

Standard definition: $n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdot \ldots \cdot n$

E.g. $3! = 1 \cdot 2 \cdot 3 = 6$

Recursive definition of factorial:

1. $0! = 1$ (basis case)
2. $n! = (n-1)! \cdot n$ (recursive/inductive step)

Example $3! = 2! \cdot 3 = 1! \cdot 2 \cdot 3 = 0! \cdot 1 \cdot 2 \cdot 3 = 1 \cdot 1 \cdot 2 \cdot 3 = 6$ (notice the necessity of providing the basis step to avoid endless recursion!)

Sum of odd naturals

$$P(n):$$

$$\sum_{i=1}^{n} 2i - 1 = 1 + 3 + 5 + ... + (2n - 1) = ?$$

Sum of odd naturals

$$P(n):$$

$$\sum_{i=1}^{n} 2i - 1 = 1 + 3 + 5 + \ldots + (2n-1) = ?$$

$$= n^2$$

Is the above equation true for all $n \in N$?

(proof by mathematical induction)

Sum of powers of 2

$$P(n):$$

$$\sum_{i=0}^{n} 2^n =$$

Sum of powers of 2

$$P(n):$$

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Sum of powers of 2

$$P(n):$$

$$\sum_{i=0}^{n} 2^n =$$

$$1 + 2 + 4 + ... + 2^n = ?$$

$$= 2^{n+1} - 1$$

Is the above equation true for all $n \in N$?

(proof by mathematical induction)

Example of inequality

$$P(n):$$

$$n < 2^n$$

Example of inequality

$$P(n):$$

$$n < 2^n$$

Is the above inequality true for all $n \in N$?

(proof by mathematical induction)

Another Example of inequality

$$P(n):$$

$$2^n < n!$$

Another Example of inequality

$$P(n):$$

$$2^n < n!$$

For which values of n is the above inequality true?

(proof by mathematical induction)

Harmonic numbers

A harmonic number H_n is defined as:

$H_n = \sum_{i=1}^{n} \frac{1}{i}$

For which values of n is the following true:

$H_{2^n} \geq 1 + \frac{n}{2}$

(proof by mathematical induction)

Example on divisibility

P(n):

$3|(n^3 - n)$

for which values of n is the above statement true?

(proof by mathematical induction)

Generalisation of De Morgan Law

Let's consider a family of subsets of some universe U: $A_i \subset U$, indexed by natural numbers $i \in N$. Let A_i' denote the complement of A_i.

P(n):

$$(\bigcap_{i=1}^{n} A_i)' = \bigcup_{i=1}^{n} A_i'$$

For which values of n is the above law true?

(proof by mathematical induction)

Proof of the generalised de Morgan Law

Basis step:
Let's start the induction from $n_0 = 2$.
$P(2)$: $(A_1 \cap A_2)' = A_1' \cup A_2'$

This is true since it is a standard de Morgan Law.

Inductive assumption:
$(\bigcap_{i=1}^{k} A_i)' = \bigcup_{i=1}^{k} A_i'$

Inductive step:
$(\bigcap_{i=1}^{k+1} A_i)' = (\bigcap_{i=1}^{k} A_i \cap A_{k+1})' = (\bigcap_{i=1}^{k} A_i)' \cup A_{k+1}' =$
$(\bigcup_{i=1}^{k} A_i') \cup (A_{k+1})' = \bigcup_{i=1}^{k+1} A_i'$

The induction step is done what completes the proof.

Example from graph theory: Number of edges in a tree

P(n): any tree having n vertices has exactly n-1 edges.

For which values of n is the above statement true?

(proof by mathematical induction)

Number of edges in a tree, cont.

Tree: a graph that is connected and does not have cycles.

Fact: each tree has at least 1 leaf (why?)

Non-numeric example: tiling of checkerboards

P(n): Each checkerboard of size $2^n \times 2^n$ with exactly 1 square removed can be tiled using L-shaped pieces covering 3 squares each.

For which values of n is the above statement true?

(proof by mathematical induction)

Strong Mathematical Induction

It is a variant of mathematical induction that makes it possible to use a stronger variant of inductive assumption:

If the following 2 conditions hold, for some predicate $P(n)$, $n \in N$:

1. $P(n_0)$ is true for some $n_0 \in N$ (**Basis step**)
2. $P(1) \wedge P(2) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)$ is true for any $k \geq n_0$ (**Inductive step**)

then: the predicate $P(n)$ is true for all $n \geq n_0$.

Strong mathematical induction is logically equivalent to standard mathematical induction (i.e. one implies another) and both are equivalent to the well-ordering of the natural numbers.

Example: number of edges of a tree

Let's now use strong induction to prove:

P(n): each n-vertex tree has exactly n-1 edges.

(proof by strong induction)

Observation: removing 1 edge from a tree results in 2 smaller trees. (because any edge is not part of any cycle)

Notice: in this kind of proof it is easier to use strong mathematical induction here than the standard one.

Prime factorisation

P(n): a number n is a product of primes

for which values of n is the above statement true?

(proof by strong induction)

Notice: it is easier to use strong mathematical induction in this proof.

Examples of Mistakes in Mathematical Induction

The typical mistakes in Mathematical Induction can be the following:

- ignoring the basis step (even if the inductive step can be done!)
- wrong induction step

Both: the basis step and the inductive step are necessary to construct a valid proof by mathematical induction.

Example of Mistake of ignoring the basis step

Prove that for all $n \in N$ the following holds:
$P(n)$: $n + 2 < n$
Let's ignore the basis step and proceed directly to the inductive step:

Inductive assumption:
$P(k)$: $k + 2 < k$.

Inductive step:
$P(k+1)$: $(k+1) + 2 = (k+2) + 1 < k + 1$

The inductive step can be proven! But $P(k)$ is not true for any k since the basis step is missing
(the basis step is not true for any $k \in N$!)

Another example of mistake

Let's prove the statement: $P(n)$: any set of n cars is of the same color (i.e. all the cars have the same color!)

Basis step (let's start from $n_0 = 1$):
$P(1)$: any set of 1 car if of the same color (true)

Inductive assumption:
$P(k)$: any set of k cars if of the same color.

Inductive step:
Let's prove $P(k+1)$: any set of k+1 cars is of the same color.

The set of the first k cars has the same color (by inductive assumption). The set of last k cars also is of the same color (again: inductive assumption). Thus, since the middle k-1 cars (2,3,...,k) are common for the two sets, all the k+1 cars have the same color.

Where is the mistake?

Why does the mathematical Induction work?

Equivalence to well ordering.

Well ordering of natural numbers:
(reminder:)

A set is well ordered if its any non-empty subset has the smallest element.

The set of natural numbers, ordered by the $\leq$ relation is well-ordered.

General properties of Natural Numbers

The following conditions[2] come from the **Peano's system of axioms of natural numbers**: ($S(n)$ denotes the **successor** function, $S(n) = n + 1$, e.g. $S(0) = 1$, $S(1) = 2$, etc.)

1. $0 \in N$
2. for any $n \in N$ it holds that $S(n) \in N$ (its successor is also in N)
3. every element of N except 0 is a successor of exactly 1 element
4. induction axiom[3]: if a set $A \subseteq N$ satisfies 2 conditions:
   - $0 \in A$
   - for any $n \in N$ the fact that $n \in S$ implies that also $S(n) \in N$

   Then it holds that $A = N$.

[2] conditions 1-3 are first-order logic the condition 4 is a second-order logic (quantifies set variable)

[3] Induction axiom is logically equivalent to the well-ordering property of natural numbers.

Why does the Mathematical Induction work?

The principle of mathematical induction is implied by the fact that the natural numbers are well-ordered.

hint: imagine the smallest element s of the set of natural numbers that do not satisfy the property $P(n)$ and the number p such that $s = S(p)$. Hence, s must be either smaller then n_0 or it would lead to a contradiction with the inductive step.

Summary

- Mathematical Induction
- Examples of numerical and non-numerical statements than can be proven by mathematical induction
- Strong Mathematical Induction
- Equivalence of Mathematical Induction with the well-ordering of the natural numbers

Example tasks/questions/problems

- Formulate the principle of mathematical induction
- Formulate the principle of strong mathematical induction
- How mathematical induction is implied by the fact that natural numbers are well ordered
- disprove or prove by mathematical induction that for any $n \in N$:
  - $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$
  - $\sum_{i=1}^{n} i^3 = (n(n+1)/2)^2$
  - $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$
  - the number of all subsets of n-element set is 2^n
  - $n^2 + n$ is always even
  - $3|(n^3 + 2n)$
  - $5|(n^5 - n)$
  - $6|(n^3 - n)$

Thank you for your attention.